

Invariance property of wave scattering through disordered media

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A fundamental insight in the theory of diffusive random walks is that the mean length of trajectories traversing a finite open system is independent of the details of the diffusion process. Instead, the mean trajectory length depends only on the system's boundary geometry and is thus unaffected by the value of the mean free path. Here we show that this result is rooted on a much deeper level than that of a random walk, which allows us to extend the reach of this universal invariance property beyond the diffusion approximation. Specifically, we demonstrate that an equivalent invariance relation also holds for the scattering of waves in resonant structures as well as in ballistic, chaotic or in Anderson localized systems. Our work unifies a number of specific observations made in quite diverse fields of science ranging from the movement of ants to nuclear scattering theory. Potential experimental realizations using light fields in disordered media are discussed.

In the biological sciences it has been appreciated for some time now that the movement of certain insects (like ants) on a planar surface can be modeled as a diffusive random walk with a given constant speed v [1–3]. Using this connection, Blanco *et al.* [4] proved that the time that these insects spend on average inside a given domain of area A and with an external boundary C is independent of the parameters entering the random walk such as, e.g., the transport mean free path (MFP) ℓ^* . Specifically, the average time t between the moments when an insect enters the domain and when it first exits it again, is given by the simple relation $\langle t \rangle = \pi A / (Cv)$. One finds that the mean length $\langle l \rangle$ of the corresponding random walk trajectories inside the domain is also constant, $\langle l \rangle = \langle t \rangle v = \pi A / C$. Similar relations also hold in three dimensions, $\langle t \rangle = 4V / (\Sigma v)$ and $\langle l \rangle = 4V / \Sigma$, where V is the volume and Σ is the external surface of a given domain. Extensions of this result exist for trajectories beginning inside the domain [5] or for the calculation of averaged residence times inside sub-domains [6]. As a generalization of the mean-chord-length theorem [7] for straight-line trajectories with an infinite MFP, this fundamental theorem has numerous applications, e.g., in the context of food foraging [8] and for the reaction rates in chemistry [9].

The surprising element of this result can be well appreciated when applied to the physical sciences and, in particular, to the transport of light or of other types of waves in scattering media. In that context it is well-known that the relevant observable quantities all do depend on ℓ^* : In the diffusive regime, the total transmission of a slab of thickness L scales with ℓ^* / L through Ohm's law, and the characteristic dwell time scales with the so-called Thouless time $L^2 / (v \ell^*)$ [10]. When considering coherent wave effects, ℓ^* also determines the width of the coherent backscattering cone in weak localization [11, 12] and drives the phase-transition from diffusive to Anderson localization [13]. An invariant quantity that does

not depend on ℓ^* would thus be highly surprising to the community involved in wave scattering through disordered media. Since, in addition, coherent effects like weak or strong (Anderson) localization clearly fall outside the scope of a diffusive random walk model, one may also expect that an invariance property simply does not exist when wave interference comes into play. As we will demonstrate here explicitly, this expectation is clearly too pessimistic. Instead, we find that an invariant time and length scale can also be defined for waves, even when they scatter non-diffusively as in the ballistic or in the Anderson localization regime. The key insight that allows us to establish such a very general relation for the mean wave scattering time is its connection to the density of states (DOS), which is the central quantity that stays invariant on a level far beyond the scope of a diffusion approximation.

To describe wave transport in a disordered scattering medium without solving the full wave equation numerically is a challenging task, which can be approached from many different angles [10, 14, 15]. As the first step, we will consider the radiative transfer equation (RTE), which describes the transport of an averaged radiation field through a disordered medium in the limit $k \ell_s \gg 1$, where $k = \omega / c = 2\pi / \lambda$ is the wave number and ℓ_s is the scattering MFP [7, 16]. In non-absorbing media as considered here the scattering MFP ℓ_s is connected to ℓ^* by the anisotropy parameter g which measures the degree of forward-scattering at a scattering event, $\ell_s = \ell^* (1 - g)$. In its standard formulation where the RTE does not include wave interference effects it should fully reproduce the predictions by Blanco *et al.* from above. However, one can enhance the scope of the RTE to include specific wave effects like the dispersion in a medium containing strongly resonant scatterers such as atomic dipoles or Mie spheres [17, 18]. In what follows, we will consider identical, but randomly placed resonant and non-absorbing dipole scatterers described by a polarizability

$\alpha(\delta) = -4\pi/k^3 [i + 2\delta/\Gamma]^{-1}$, with $\delta = \omega - \omega_0$ the detuning with respect to the resonance frequency ω_0 , and Γ the linewidth (modelling losses due to scattering only)[58]. This specific expression is valid close to the resonance (i.e. $\delta \ll \omega_0$) and ensures energy conservation (i.e. the optical theorem is fulfilled). In this case and for a dilute system such that $\mathcal{N}\lambda^3 \ll 1$, where \mathcal{N} is the density of scatterers, a dispersive form of the RTE can be derived from the Bethe-Salpeter equation (an exact equation for the spatio-temporal auto-correlation function of the electric field) [18]:

$$\left[-\frac{i\Omega}{c} + \mathbf{u} \cdot \nabla_{\mathbf{r}} + \mu_e(\delta, \Omega) \right] I(\mathbf{u}, \mathbf{r}, \delta, \Omega) = \frac{1}{4\pi} \mu_s(\delta, \Omega) \int I(\mathbf{u}', \mathbf{r}, \delta, \Omega) d\mathbf{u}' \quad (1)$$

where $d\mathbf{u}'$ stands for integration over the solid angle. The specific intensity $I(\mathbf{u}, \mathbf{r}, \delta, \tau)$ (also called spectral radiance) describes the radiative flux at position \mathbf{r} , along direction \mathbf{u} , at frequency δ and at time τ . It is defined from the Wigner transform of the electric field. Equation (1) is Fourier transformed with respect to τ (with Ω being the conjugate Fourier variable). The expressions for the extinction and scattering MFP are given by $\mu_e(\delta, \Omega) = -i\mathcal{N}k/2[\alpha(\delta + \Omega/2) - \alpha^*(\delta - \Omega/2)]$ and $\mu_s(\delta, \Omega) = \mathcal{N}k^4/(4\pi)\alpha(\delta + \Omega/2)\alpha^*(\delta - \Omega/2)$. The Boltzmann scattering MFP $\ell_s(\delta) = \ell_0 [1 + 4\delta^2/\Gamma^2]$, with $\ell_0 = [4\pi\mathcal{N}/k^2]^{-1}$ the value at resonance ($\delta = 0$), can be changed by varying the detuning δ or the linewidth Γ . Note that the condition $\mathcal{N}\lambda^3 \ll 1$ is then equivalent to $k\ell_s \gg 1$.

On this basis we can evaluate the average time spent by light trajectories inside the medium by calculating the weighted temporal average $\langle t(\delta) \rangle = \int \tau \phi_{\text{out}}(\delta, \tau) d\tau / \int \phi_{\text{out}}(\delta, \tau) d\tau$, where the weighting function $\phi_{\text{out}} = \int_{\Sigma} \int_{2\pi} I(\mathbf{u}, \mathbf{r}, \delta, \tau) \mathbf{u} \cdot \mathbf{n} d\mathbf{u} d^2\mathbf{r}$ is the outgoing flux at time τ , Σ is the medium boundary and \mathbf{n} the outward normal. In frequency domain, this expression can be cast in the following compact form (see appendix B):

$$\langle t(\delta) \rangle = \frac{-i}{\phi_{\text{out}}(\delta, \Omega = 0)} \left. \frac{\partial \phi_{\text{out}}(\delta, \Omega)}{\partial \Omega} \right|_{\Omega=0}, \quad (2)$$

where we define $t = 0$ as the time when the incident flux enters the medium. This expression is also convenient for a numerical computation of $\langle t(\delta) \rangle$ based on Eq. (1). In the numerical simulation, we consider a 3D slab geometry of length L with on-resonance optical thickness $b_0 = L/\ell_0$, illuminated by an isotropic and uniform specific intensity on its left interface only, see Fig. 1(a). This corresponds to a situation where the incident specific intensity does not depend on the point and direction of incidence (Lambert's cosine law is satisfied). Using a Monte-Carlo scheme [18], we solved Eq. (1) without approximation and obtained the results plotted in Fig. 2. By tuning the linewidth Γ of the scatterers, we can ei-

ther simulate a non-resonant medium in which the intensity spends most of the time between the scatterers ($\Gamma\ell_0 \gg c$) or a resonant medium where the transport time is dominated by intensity trapping inside the scatterers ($\Gamma\ell_0 \ll c$).

In the non-resonant case, see Fig. 2(a), we recover the results by Blanco *et al.* [4] and find an average time $\langle t(\delta) \rangle$ that is independent on the scattering properties of the medium (i.e. independent of the detuning δ that determines the scattering properties in the present RTE calculation). Moreover, we clearly see that the times associated to the reflected and transmitted parts of the outgoing flux, that can be computed separately, strongly depend on δ , showing that the invariance of the average time $\langle t(\delta) \rangle$ results from a delicate balance between reflection and transmission (in both their intensities and time delays), as illustrated by light trajectories displayed in Fig. 1(b),(c). Also note that by varying the detuning δ from 0 to 2 in Fig. 2(a), we effectively perform a crossover from the diffusive to the single scattering regime. In the latter ($\delta \gtrsim 1.5$), the invariance of $\langle t(\delta) \rangle$ follows directly from the mean-chord-length theorem [7].

In case of a resonant medium, see Fig. 2(b), the situation is substantially different. The average time $\langle t(\delta) \rangle$ exhibits a significant dependence on δ , and therefore on the scattering properties of the medium. As this result clearly falls outside the scope of the invariance relation derived by Blanco *et al.* [4], the question arises, whether a new quantity can be defined that remains invariant even in the limit of strongly dispersive scatterers. To address this issue, we rewrite the average time $\langle t(\delta) \rangle$ in Eq. (2) as the ratio of the total energy U stored in the system and the outgoing flux ϕ_{out} ,

$$\langle t(\delta) \rangle = \frac{U(\delta, \Omega = 0)}{\phi_{\text{out}}(\delta, \Omega = 0)}. \quad (3)$$

As specified in more detail in the supplemental material, this relation measures $\langle t(\delta) \rangle$ as the time the stored energy U takes to flow out of the medium with flux $\phi_{\text{out}} = -\phi_{\text{in}}$ (in stationary processes without absorption or gain, the incoming and outgoing fluxes are balanced) [19]. Expressing ϕ_{out} and U in terms of the specific intensity $\phi_{\text{out}}(\delta, \Omega) = \int_{\Sigma} \int_{2\pi} I(\mathbf{u}, \mathbf{r}, \delta, \Omega) \mathbf{u} \cdot \mathbf{n} d\mathbf{u} d^2\mathbf{r}$ and $U(\delta, \Omega) = v_E^{-1}(\delta) \int_V \int_{4\pi} I(\mathbf{u}, \mathbf{r}, \delta, \Omega) d\mathbf{u} d^3\mathbf{r}$, where $v_E(\delta)$ is the energy (or transport) velocity [17], we obtain

$$\langle t(\delta) \rangle = \left[\frac{1}{v_E(\delta)} \int_V \int_{4\pi} I(\mathbf{u}, \mathbf{r}, \delta, \Omega = 0) d\mathbf{u} d^3\mathbf{r} \right] \times \left[\int_{\Sigma} \int_{2\pi} I(\mathbf{u}, \mathbf{r}, \delta, \Omega = 0) \mathbf{u} \cdot \mathbf{n} d\mathbf{u} d^2\mathbf{r} \right]^{-1}. \quad (4)$$

In this expression V and Σ are the volume and the external boundary of the medium, and \mathbf{n} is the outward normal. For a uniform and isotropic illumination on the surface (as assumed here), the specific intensity is uniform, isotropic and independent on detuning inside the

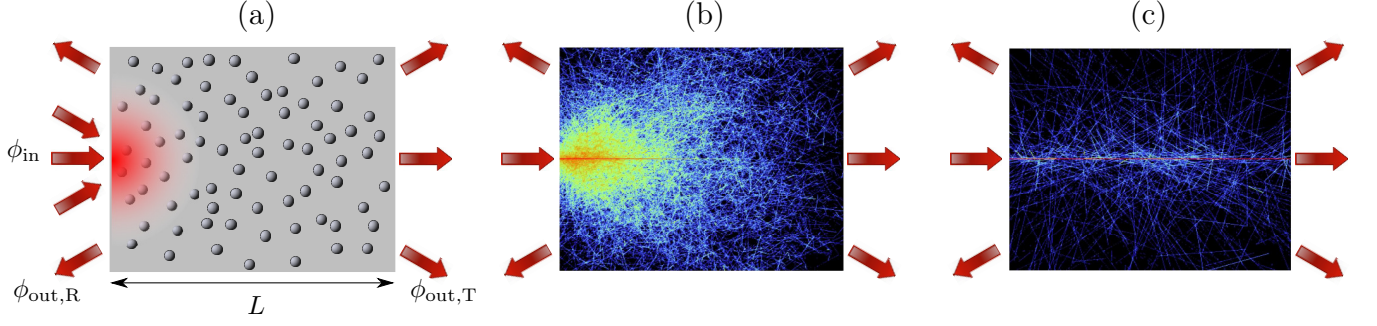


Figure 1: Sketch of the system and light trajectories. (a) Geometry of the 3D slab of length L investigated numerically using the RTE. ϕ_{in} is the incident flux and $\phi_{\text{out,R}}$, $\phi_{\text{out,T}}$ are the reflected and transmitted fluxes, respectively ($\phi_{\text{out}} = \phi_{\text{out,R}} + \phi_{\text{out,T}}$). (b),(c) Projections of light trajectories inside the system in the case of normal incidence illumination at a specific point on the system boundary (see incoming arrow) for two optical thicknesses (b) $b = 10$ and (c) $b = 0.59$.

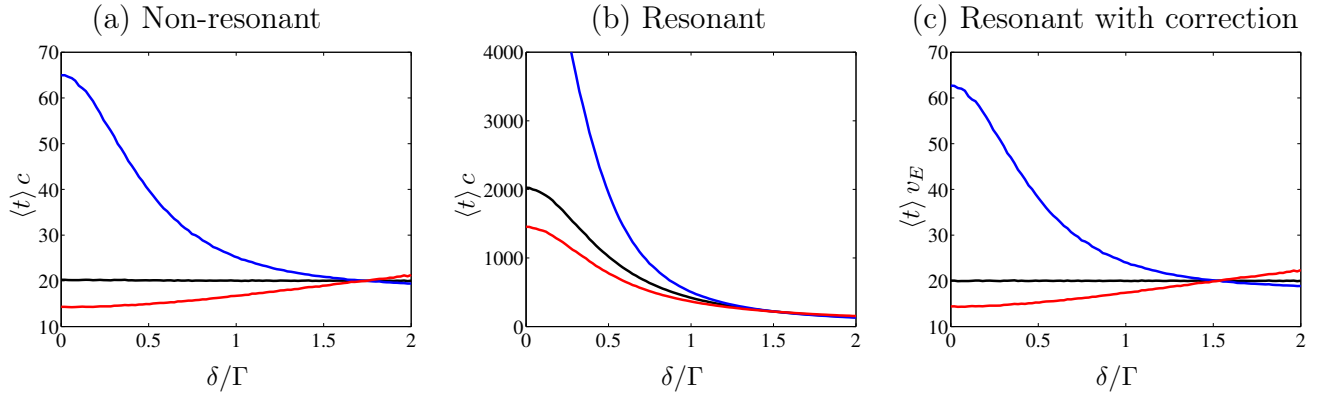


Figure 2: Ensemble-averaged length $\langle l(\delta) \rangle$ of light trajectories as obtained numerically using the RTE for (a) non-resonant and (b),(c) resonant scatterers in a 3D slab of width $L = 10$ and optical thickness at resonance $b_0 = 10$. Black/blue/red lines depict the values for all/transmitted/reflected trajectories, corresponding to the fluxes $\phi_{\text{out}}/\phi_{\text{out,T}}/\phi_{\text{out,R}}$ in Fig. 1. The values for $\langle l(\delta) \rangle$ were determined through the average time $\langle t \rangle$ multiplied by the speed of light c in panels (a),(b) and by the energy velocity v_E in panel (c). The renormalization with the energy velocity v_E in the resonant case (c) yields the same universal value $\langle t \rangle v_E = 2L$ as obtained by Blanco *et al.* for the non-resonant case $\langle t \rangle c = 2L$, see (a). For $\delta = 0$ and for $\delta = 2\Gamma$, the optical thickness is $b = 10$ and $b = 0.59$, respectively, such that the above results range from the diffusive to the single-scattering regime.

medium (a particular case of such a situation is black-body radiation) [14]. As a result, Eq. (4) can be drastically simplified into $\langle t(\delta) \rangle = 4V/[\Sigma v_E(\delta)]$, which for a slab of thickness L gives $\langle t(\delta) \rangle = 2L/v_E(\delta)$. This result turns out to be strikingly similar to the invariance relation derived by Blanco *et al.* [4], the only difference being that in resonant media the dispersive form of the energy velocity $v_E(\delta)$ comes into play. The expression of the energy velocity for resonant scatterers can be determined explicitly [17], and takes the following form (see appendix A)

$$v_E(\delta) = \left[\frac{1}{c} + \frac{1}{\Gamma \ell_s(\delta)} \right]^{-1}. \quad (5)$$

The energy velocity allows us to introduce an invariant length scale, $\langle l \rangle = \langle t(\delta) \rangle v_E = 4V/\Sigma$, which is independent on the scattering properties of the medium for both

resonant and non-resonant scattering (in the latter the energy velocity simply reduces to the constant velocity entering the random walk formalism). To prove the correctness of this result, we plot the average length $\langle l \rangle$ in Fig. 2(c) as obtained by renormalizing the numerical results for $\langle t(\delta) \rangle$ in Fig. 2(b) with the analytical expression (5) of the transport velocity v_E . We find that the resulting curve for $\langle l(\delta) \rangle = \langle t(\delta) \rangle v_E$ is, indeed, independent on the detuning δ , with a constant value $\langle l \rangle = 2L$. This result is all the more remarkable as the average lengths associated with either the transmitted or the reflected part of the flux display a strong dependence on the scattering properties in the same regime. This again shows that the invariance of the average length $\langle l \rangle$ results from a subtle balance between reflection and transmission.

Whereas the above extension of the RTE allowed us to find a new invariant quantity for the case of scattering in a disordered medium with resonant scatterers, the ansatz

of the RTE itself is intrinsically restricted to the limit $k\ell_s \gg 1$. The opposite limit, where the wave length λ is comparable to or even larger than the mean free path ℓ_s , is thus not covered by our foregoing considerations. As in this strongly scattering limit wave interference can lead to a complete halt of wave diffusion in terms of Anderson localization, the question arises whether localization will lead to a deviation from the above invariance property or not. One could expect such a deviation, e.g., on the grounds that localization prevents scattering states to explore the entire scattering volume V of the system. Correspondingly, the volume V and the surface Σ appearing in the invariance relation $\langle t(\delta) \rangle = 4V/[\Sigma v_E(\delta)]$ might then have to be rescaled with the localization length ξ .

To explore this question in detail we will now work with the full wave equation in two dimensions which, for stationary light scattering, is given in terms of the Helmholtz equation

$$\left[\Delta + n(x, y)^2 k^2 \right] \psi(x, y) = 0. \quad (6)$$

The linear dispersion $k = \omega/c$ will allow us to use k and ω interchangeably. In the situations we study here, the disorder scattering is induced by the spatial variations of the static refractive index $n(x, y)$. To evaluate the dwell time of a stationary scattering eigenstate of this equation (with well-defined wave number k) inside a given spatial region one can conveniently use the so-called Wigner-Smith time-delay operator [59]

$$Q(\omega) = -i S^{-1} \frac{dS}{d\omega}, \quad (7)$$

originally introduced by Wigner in nuclear scattering theory [20] (and extended by Smith to multi-channel scattering problems [21]). Here the ω -dependent scattering matrix S , evaluated at the external boundary C of the considered region, contains all the complex transmission and reflection amplitudes that connect in- and outgoing waves in a suitable mode basis. To obtain also here the average time associated with wave scattering we take the trace of Q and divide by the number $N(\omega)$ of incoming scattering channels, $\langle t(\omega) \rangle = \text{Tr}[Q(\omega)]/N(\omega)$.

To evaluate the average time $\langle t(\omega) \rangle$ from above, we performed numerical simulations on a two-dimensional scattering region of rectangular shape, attached to perfect semi-infinite waveguides on the left and right (see illustrations in the lower panels of Fig. 3). Accordingly, the correct number of scattering channels $N(\omega)$ is given by the total number of flux-carrying modes in both waveguides. Impenetrable and non-overlapping circular scatterers are randomly placed inside the scattering region and in between them the refractive index is kept constant, $n(x, y) = 1$. The scattering matrix and the corresponding scattering states for this system are calculated by solving the Helmholtz Eq. (6) on a finite-difference grid, using the advanced modular recursive Green's function method [22, 23]. In Fig. 3 we display our numerical results for different degrees of disorder: In subfigure (a), we

show the results obtained for an empty scattering region, corresponding to the ballistic transport regime. In subfigure (b), the case with altogether 13 scatterers is shown, for which already a strong reduction of transmission is observed. The distribution of the transmission eigenvalues $P(\tau)$ follows here very well the predictions of Random Matrix Theory for the regime of chaotic scattering (see appendix D). Finally, in subfigure (c), we increased the degree of disorder even more (placing altogether 211 scatterers) such as to enter the regime of Anderson localization. Here the distribution of transmission eigenvalues agrees very well with the predictions for the case when Anderson localization suppresses all but a single transmission eigenchannel (see appendix D) [24, 25]. To make all three cases easily comparable with each other, the different geometries all have the same scattering area A , which, for ballistic scattering is the entire rectangular region between the leads, whereas for the other two cases the area occupied by the impenetrable scatterers is not part of A .

Based on the above identification of the different transport regimes that our model system can be in, we investigate now the corresponding results for the average time $\langle t(\omega) \rangle$ which we get for each of these limits (see panels in Fig. 3). In the ballistic limit [see panel (a)], we see that the average time, plotted as a function of the incoming wavenumber k , shows pronounced periodic enhancements around the random walk prediction by Blanco *et al.* $\langle t \rangle = \pi A/(Cv)$. The peaks of these fluctuations can be identified with those positions in $k = k_n = n\pi/d$, where a new transverse mode opens up in the waveguide of width d . To understand why these mode openings cause an increase in the scattering dwell time, we resort to a fundamental connection between the average dwell time $\langle t \rangle$ and the density of states (DOS) $\rho(k)$. This relation, $\rho(k) = N(k)c\langle t(k) \rangle/(2\pi) = c\text{Tr}(Q)/(2\pi)$, was first put forward by Birman, Krein, Lyuboshitz and Schwinger in the context of quantum electrodynamics and nuclear scattering theory and has meanwhile been used in a variety of different contexts [26–36]. Since, in the ballistic regime, each individual incoming mode corresponds to a one-dimensional scattering channel with, correspondingly, an associated square root singularity in the DOS, $\rho_n(k) = [L/(2\pi)] k/\sqrt{k^2 - k_n^2}$ for $k > k_n$, we can successfully explain the observed oscillations as coming from the successive openings of new waveguide modes. Evaluating the total DOS based on a sum of individual mode contributions, $\rho(k) = \sum_n^N \rho_n(k)$, and using the above connection to the average time yields identical results to those shown in Fig. 3(a). This demonstration also allows us to show that the time, averaged over an interval of k that is larger than the distance between successive mode openings, converges exactly to the prediction by Blanco *et al.* Quite remarkably, we find in this sense that the estimate from the mean-chord-length theorem and, correspondingly, the random walk prediction also holds, on average, for ballistic wave scattering in a system without any disorder.

Moving next to the disordered system in Fig. 3(b), we see that the presence of the disorder strongly reduces the above mode-induced fluctuations, leaving the frequency-average value of time unchanged. To explain this result, the DOS clearly needs to be estimated differently here than in the ballistic case of uncoupled waveguide modes. Also, since the disorder leads to system and frequency specific fluctuations of the DOS, we are looking here for an estimate for the ensemble and frequency averaged DOS. To obtain this quantity, we invoke a result first put forward by Weyl in 1911 [37], who estimated that the average DOS in the asymptotic limit of $\omega \rightarrow \infty$ satisfies the following universal law, $\rho(\omega) = A\omega/(2\pi c^2)$, now called the Weyl law [38]. Putting this estimate into the formula relating the average time with the average DOS, we obtain $\langle t(\omega) \rangle = 2\pi\rho(\omega)/N(\omega) = A\omega/[c^2N(\omega)]$. The ω -dependent number of incoming channels is given as an integer-valued step-function $N(\omega) = \lfloor 2\omega d/(c\pi) \rfloor$. When smoothing over the steps in this function, i.e., $N(\omega) \approx 2\omega d/(c\pi) - 0.5$, we arrive at the result $\langle t(\omega) \rangle = 2\pi\rho(\omega)/N(\omega) \approx \pi A/(2dc) = \pi A/(Cc)$. This relation, which is very accurately satisfied by our numerical results, thus confirms the validity of the diffusive random walk prediction by Blanco *et al.* also for disordered wave scattering. Because the above relation for the average dwell time is notably independent of ℓ^* , transmission and reflection times for waves, which do strongly depend on ℓ^* , need to fully counter-balance each other.

Does this invariance of the average scattering time also persist in the strongly scattering limit, when Anderson localization sets in? Our numerical results shown for this case in Fig. 3(c) display a small but apparently systematic frequency dependence of the average time $\langle t(\omega) \rangle$ which increasingly deviates from the result by Blanco *et al.* for decreasing frequencies ω . Since the numerical calculations are very challenging and the frequency derivative appearing in Eq. (7) can reach very large values for highly localized scattering states, we first tested the accuracy of our simulations by evaluating $\langle t(\omega) \rangle$ also through explicit dwell time calculations. In analogy to Eq. (3), the expression for the dwell time in case of the Helmholtz Eq. (6) is given by $t_m = \int_A \psi_m^* n^2 \psi_m d^2\mathbf{r} / \phi_{m,\text{in}}$, where ψ_m is the wave function of the m -th scattering channel and $\phi_{m,\text{in}}$ is the corresponding total (stationary) incoming flux. The average dwell time is then given by $\langle t(\omega) \rangle = \Sigma_m^N t_m(\omega) / N(\omega)$. The results obtained in this way are practically indistinguishable from Fig. 3(c). To explain this robust deviation from the result by Blanco *et al.* [4], we thus have a more careful look on the Weyl estimate which, in addition to the leading order term which we used above, also features a next-order correction proposed by Weyl [38, 39], $\rho(\omega) = [A\omega/c^2 + (C - B)/(2c)]/(2\pi)$. This correction involves not only the scattering area A , but also the internal boundary of the scattering region B which is notably different from the external boundary C through which waves can scatter in and out. The internal boundary B in case of our waveguide system under study is given by

$B = 2L + B_o$, where B_o is given by the total circumference of the scatterers. The open boundary conditions along the external boundary C were approximated with Neumann boundary conditions, which contribute with the opposite sign as the Dirichlet boundary conditions on the surface of the waveguide and of the scatterers. In systems with a small boundary-to-area ratio this next-order correction of the Weyl law is negligible. Since, however, the number of scatterers which we have placed inside the system (from 0 in the ballistic case, to 13 in the chaotic case, to 211 in the localized case) increases this ratio, the additional boundary term in the Weyl law may become important here. To check this explicitly, we re-evaluate the expression for the average dwell time $\langle t(\omega) \rangle$ from above when adding this correction, leading us to

$$\langle t(\omega) \rangle = \frac{1}{c^2 N(\omega)} \left[A\omega + \frac{(C - B)}{2} c \right]. \quad (8)$$

A comparison of this analytical formula with the numerical results, see Fig. 3(c), yields excellent agreement and indicates that the observed deviation from the prediction by Blanco *et al.* stems from the comparatively large boundary of the many small scatterers which we placed inside the scattering region. We emphasize at this point that this correction to the Blanco estimate only contains the boundary values B and C as additional input and remains entirely independent of any quantities that characterize the scattering process itself, like ℓ^* or the localization length ξ . This insight is of considerable importance, since it means that Eq. (8) defines a new invariant quantity that is independent of the scattering regime we are in and thus accurately matches our numerical results for the average time in the ballistic, chaotic and localized limit. This invariant quantity for waves deviates from the prediction by Blanco *et al.* [4] only through an additional term originating in the fact that waves feel the boundary of a scattering region already when being close to it on a scale comparable with the wave length. We speculate that additional wave corrections to the result by Blanco and Fournier may arise when waves have access to a larger scattering area A than classical particles through the process of tunneling.

In summary, we have derived a universal invariance property for wave transport through disordered media. The invariance of the averaged path length or averaged time spent by a wave in an open finite medium has been established based on scattering theory. In the appropriate limit of diffusive and non-resonant media, the random-walk picture is recovered, and the result coincides with the expression of the averaged path-length initially established by Blanco *et al.* [4]. Our work confers to this invariance property a degree of universality that extends its implications far beyond applications of random-walk theory. This extension to waves opens up new possible applications in optics, acoustics, seismology, or radiofrequency technologies, where propagation in complex media is the subject of intense research [40]. Indeed, in the context of wave transport through disor-

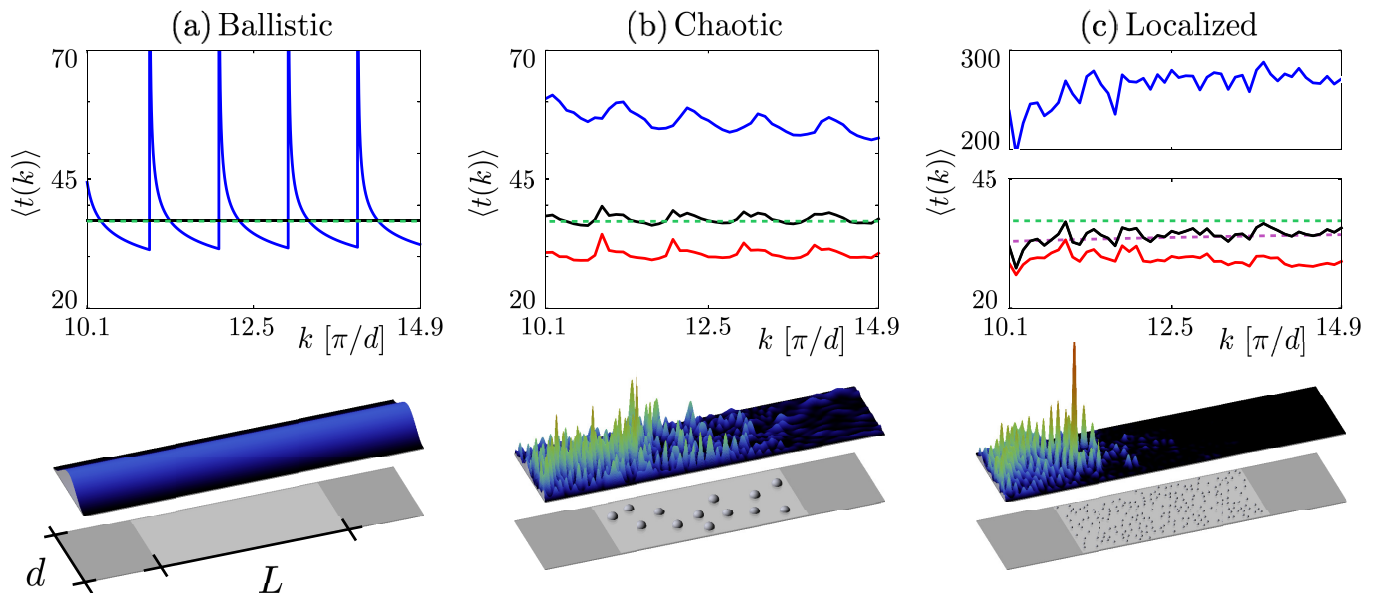


Figure 3: Total average dwell time $\langle t(k) \rangle$ (black line), transmission delay time (blue line), and reflection delay time (red line) for (a) ballistic scattering through a clean waveguide as well as for (b) chaotic scattering through a disordered waveguide with 13 circular obstacles of radius $r = 0.06d$ and (c) Anderson localized transport through a disordered waveguide with 211 obstacles of $r = 0.015d$ (see appendix C for a definition of transmission and reflection delay times). The geometrical parameters were chosen such that all three waveguides have the same width d and the same effective scattering area $A = 2.35d^2$. The wavenumber was scanned between $k = 10.1\pi/d$ and $k = 14.9\pi/d$ in all three cases. For the clean waveguide in (a) the transmission is perfect, thus the reflection times are strictly zero. The average for the total dwell time (black line) is taken here over the entire wavenumber interval shown and coincides with the estimate of Blanco *et al.* $\langle t(k) \rangle = \pi A / (Cc)$ (green dashed line). For the disordered systems in (b),(c), the averages were taken over (b) 250 and (c) 2500 different random configurations, respectively. Whereas for the chaotic scattering case (b) the results for the average dwell time agree well with the random walk prediction (dashed green line), a systematic deviation is observed for the case of strong disorder (c). Here, very good agreement is found with the estimate for the average dwell time according to the corrected Weyl estimate, Eq. (8) (purple dashed line). In the lower panels, the intensity of wave functions injected in the lowest-order mode is shown for a specific configuration of scatterers (see grey spheres) embedded in the scattering area (light grey domain in the middle). The flux is incoming from the left and can be transmitted (to the right) or reflected (to the left) through the perfect waveguides attached on both sides (see dark grey areas).

dered media, most spatial or temporal observables scale with ℓ^* , and the invariance property derived in this work is particularly counterintuitive and rich in implications. It should find applications in imaging, communications, or light delivery, for instance to generate enhanced light-matter interaction within a certain volume by controlled light deposition, or to design specific structures to enhance light harvesting for solar cells [41, 42]. Consider here for example that the above invariance property allows us to estimate the time that waves need to transit through a given medium based on a measurement of only the reflected portion of incoming waves and an *a priori* knowledge of the sample geometry. Particularly intriguing in our eyes is the possibility to get access, through Eq. (8), to the internal surface B of scatterers embedded in a scattering medium through a time-resolved transport experiment. Such an approach could go as far as to measure the fractal dimension of the scatterer surface by linking our results with the Berry-Weyl conjecture [43, 44].

An extension of our findings to media with gain and loss [45, 46] should also be of interest, both from a theo-

retical and an applied standpoint. Our study should also be very relevant to the field of wave control, which has recently emerged as a powerful paradigm for light manipulation and delivery in complex media [47], showing for instance that suitably shaped wavefronts can deliver light at a specific time and position [48–50]. Finally, let us point out that although we only studied here 3D slab and 2D waveguide geometries with uncorrelated disorder, the invariance property, thanks to its connection to the DOS, is very general and should apply to a wide range of geometries and excitation strategies, as well as to non-uniform scattering properties, biological tissues, and correlated disorder, from partially ordered to entirely ordered system such as Levy glasses or photonic crystals [51, 52]. An experimental demonstration of the discussed invariance property should be within reach, in particular in optics where time-resolved techniques and sensitive detectors are available.

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Appendix A: Transport equation and energy velocity

We recall that the transport equation for a resonant scattering system is given by [18]

$$\left[-\frac{i\Omega}{c} + \mathbf{u} \cdot \nabla_{\mathbf{r}} + \mu_e(\omega, \Omega) \right] I(\mathbf{u}, \mathbf{r}, \omega, \Omega) = \frac{1}{4\pi} \mu_s(\omega, \Omega) \int I(\mathbf{u}', \mathbf{r}, \omega, \Omega) d\mathbf{u}' \quad (\text{A1})$$

where I is the specific intensity, proportional to the radiative flux at position \mathbf{r} , in direction \mathbf{u} , at frequency ω and at time τ (Ω in frequency domain). c is the speed of light in vacuum. $\mu_e(\omega, \Omega)$ and $\mu_s(\omega, \Omega)$ are coefficients given by

$$\mu_e(\omega, \Omega) = \frac{-i\mathcal{N}k}{2} \left\{ \alpha \left(\omega + \frac{\Omega}{2} \right) - \alpha^* \left(\omega - \frac{\Omega}{2} \right) \right\} \quad (\text{A2})$$

$$\text{and } \mu_s(\omega, \Omega) = \frac{\mathcal{N}k^4}{4\pi} \alpha \left(\omega + \frac{\Omega}{2} \right) \alpha^* \left(\omega - \frac{\Omega}{2} \right) \quad (\text{A3})$$

where α is the polarizability of a point scatterer (dipole) and \mathcal{N} the density. To deal with resonant scatterers, we have chosen to write α in the form

$$\alpha(\omega) = \frac{-4\pi}{k^3} \frac{1}{i + 2(\omega - \omega_0)/\Gamma} \quad (\text{A4})$$

where $k = \omega/c$. This expression fulfills the optical theorem (energy conservation), no losses by absorption are present. Defining the detuning by $\delta = \omega - \omega_0$, the scattering length is thus given by

$$\ell(\delta) = \ell_0 \left[1 + \frac{4\delta^2}{\Gamma^2} \right] \quad (\text{A5})$$

where $\ell_0 = [4\pi\mathcal{N}/k_0^2]^{-1}$ is the scattering length at the resonant frequency ω_0 .

Integrating Eq. (A1) over the directions (first moment), it is possible to derive a conservation equation linking the energy density u and the radiative flux vector ϕ defined as follows:

$$u(\mathbf{r}, \omega, \Omega) = \frac{1}{v_E} \int I(\mathbf{u}, \mathbf{r}, \omega, \Omega) d\mathbf{u}, \quad (\text{A6})$$

$$\phi(\mathbf{r}, \omega, \Omega) = \int I(\mathbf{u}, \mathbf{r}, \omega, \Omega) \mathbf{u} d\mathbf{u}. \quad (\text{A7})$$

We obtain:

$$\left[-\frac{i\Omega}{c} + \{\mu_e(\omega, \Omega) - \mu_s(\omega, \Omega)\} \right] v_E u(\mathbf{r}, \omega, \Omega) + \nabla_{\mathbf{r}} \cdot \phi(\mathbf{r}, \omega, \Omega) = 0. \quad (\text{A8})$$

To identify with a conservation equation of the form

$$-i\Omega u(\mathbf{r}, \omega, \Omega) + \nabla_{\mathbf{r}} \cdot \phi(\mathbf{r}, \omega, \Omega) = 0, \quad (\text{A9})$$

the energy velocity should read

$$\frac{1}{v_E(\omega, \Omega)} = \frac{1}{c} + \frac{i}{\Omega} \{\mu_e(\omega, \Omega) - \mu_s(\omega, \Omega)\}. \quad (\text{A10})$$

Taking the limit $\Omega \rightarrow 0$, we finally obtain

$$\boxed{\frac{1}{v_E(\delta)} = \frac{1}{c} + \frac{1}{\Gamma \ell(\delta)}}. \quad (\text{A11})$$

Appendix B: Transport equation and average time

The average time is defined by

$$\langle t(\delta) \rangle = \langle t_{\text{out}}(\delta) \rangle - \langle t_{\text{in}}(\delta) \rangle \quad (\text{B1})$$

where the incoming and outgoing average times are given by

$$\langle t_{\text{in}}(\delta) \rangle = \frac{\int \tau \phi_{\text{in}}(\delta, \tau) d\tau}{\int \phi_{\text{in}}(\delta, \tau) d\tau} \quad (\text{B2})$$

$$\langle t_{\text{out}}(\delta) \rangle = \frac{\int \tau \phi_{\text{out}}(\delta, \tau) d\tau}{\int \phi_{\text{out}}(\delta, \tau) d\tau} \quad (\text{B3})$$

and $\phi_{\text{in,out}}(\delta, \tau)$ are the input/output fluxes at time τ and for a detuning δ . In frequency domain, this reads

$$\boxed{\langle t_{\text{in,out}}(\delta) \rangle = \frac{-i}{\phi_{\text{in,out}}(\delta, \Omega=0)} \frac{\partial \phi_{\text{in,out}}(\delta, \Omega)}{\partial \Omega} \Big|_{\Omega=0}}. \quad (\text{B4})$$

By integrating Eq. (A9) over the volume of the system we get

$$i\Omega \int_V u(\mathbf{r}, \delta, \Omega) d^3\mathbf{r} = \int_V \nabla_{\mathbf{r}} \cdot \phi(\mathbf{r}, \delta, \Omega) d^3\mathbf{r} \quad (\text{B5})$$

and using the divergence theorem we find

$$i\Omega \int_V u(\mathbf{r}, \delta, \Omega) d^3\mathbf{r} = \int_{\Sigma} \phi(\mathbf{r}, \delta, \Omega) \cdot \mathbf{n} d^2\mathbf{r} = \phi(\delta, \Omega) = \phi_{\text{in}}(\delta, \Omega) + \phi_{\text{out}}(\delta, \Omega). \quad (\text{B6})$$

As the system is not absorbing, the stationary outgoing flux is given by $\phi_{\text{out}}(\delta, \Omega = 0) = -\phi_{\text{in}}(\delta, \Omega = 0)$ and the Taylor expansion of the fluxes writes

$$\phi_{\text{in,out}}(\delta, \Omega) \sim \phi_{\text{in,out}}(\delta) + \Omega \left. \frac{\partial \phi_{\text{in,out}}(\delta, \Omega)}{\partial \Omega} \right|_{\Omega=0}. \quad (\text{B7})$$

Thus, the total stationary energy inside the system writes

$$\int_V u(\mathbf{r}, \delta, \Omega = 0) d^3\mathbf{r} = -i \left. \frac{\partial \phi_{\text{in}}(\delta, \Omega)}{\partial \Omega} \right|_{\Omega=0} - i \left. \frac{\partial \phi_{\text{out}}(\delta, \Omega)}{\partial \Omega} \right|_{\Omega=0} \quad (\text{B8})$$

and the average time becomes

$$\langle t(\delta) \rangle = \frac{U(\delta, \Omega = 0)}{\phi_{\text{out}}(\delta, \Omega = 0)} \quad (\text{B9})$$

where U is the total energy stored within the system. Using the definition of the energy density, we find that the average time renormalized by the energy velocity is given by

$$\langle t(\delta) \rangle v_E = \left[\int_{\Sigma} \int_{2\pi} I(\mathbf{u}, \mathbf{r}, \omega, \Omega = 0) \mathbf{u} \cdot \mathbf{n} d\mathbf{u} d^2\mathbf{r} \right]^{-1} \times \int_V \int_{4\pi} I(\mathbf{u}, \mathbf{r}, \omega, \Omega = 0) d\mathbf{u} d^3\mathbf{r}. \quad (\text{B10})$$

This quantity can be seen as the average length of the random walk process inside the system and as it depends only on the specific intensity for a given frequency, this is the right quantity that should be conserved whatever the detuning. Indeed, if we illuminate the system with an isotropic specific intensity I_0 at each point of the boundary, the only solution is $I = I_0$ inside the system and the average length reads

$$\langle t(\delta) \rangle v_E = \frac{4\pi V I_0}{\pi \Sigma I_0} = \frac{4V}{\Sigma} \quad (\text{B11})$$

where Σ and V are the surface and the volume of the system, respectively.

Appendix C: Average Transmission and Reflection Delay Times

The average total delay time in scattering systems described by a wave equation such as the Helmholtz equation can conveniently be written as the trace of the time delay operator Q divided by the total number of open scattering channels N (see also main text). Using the scattering amplitudes stored in the scattering matrix S , we can rewrite the corresponding expression as follows

$$\langle t \rangle = \frac{1}{N} \text{Tr}(Q) = \frac{1}{N} \left(\sum_{m,n} |S_{mn}|^2 \frac{d\varphi_{mn}}{d\omega} \right), \quad (\text{C1})$$

where $S_{mn} = |S_{mn}| e^{i\varphi_{mn}}$ is the complex scattering amplitude connecting the n -th incoming and the m -th outgoing channel. For the 2-port systems we study, the scattering matrix can formally be decomposed into four distinct blocks,

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \quad (\text{C2})$$

the matrices r and t contain the elements associated with reflection and transmission for injection through the left waveguide, respectively. The primed quantities contain the corresponding elements for injection from the right. Using this division into reflected and transmitted parts, we can define the average total transmission $\langle T_{\text{tot}} \rangle$ and reflection $\langle R_{\text{tot}} \rangle$ according to

$$\begin{aligned} \langle T_{\text{tot}} \rangle &= \frac{1}{N} \left(\sum_{m,n} |t_{mn}|^2 + |t'_{mn}|^2 \right) \\ &= 1 - \frac{1}{N} \left(\sum_{m,n} |r_{mn}|^2 + |r'_{mn}|^2 \right) = 1 - \langle R_{\text{tot}} \rangle. \end{aligned} \quad (\text{C3})$$

The effective number of transmitting channels then evaluates to $N_T = \langle T_{\text{tot}} \rangle N$ and analogously the effective number of reflected channels is $N_R = \langle R_{\text{tot}} \rangle N$. Very similar to Eq. (C1), we can then finally define the average transmission time $\langle t_T \rangle$ and the average reflection time $\langle t_R \rangle$ as

$$\langle t_T \rangle = \frac{1}{N_T} \left(\sum_{m,n} |t_{mn}|^2 \frac{d\varphi_{mn}^t}{d\omega} + |t'_{mn}|^2 \frac{d\varphi_{mn}^{t'}}{d\omega} \right), \quad (\text{C4})$$

and

$$\langle t_R \rangle = \frac{1}{N_R} \left(\sum_{m,n} |r_{mn}|^2 \frac{d\varphi_{mn}^r}{d\omega} + |r'_{mn}|^2 \frac{d\varphi_{mn}^{r'}}{d\omega} \right), \quad (\text{C5})$$

with, e.g., $r_{mn} = |r_{mn}| e^{i\varphi_{mn}^r}$ denoting a complex reflection amplitude from left to left. Note that the properly weighted sum of the times (C4) and (C5) add up to the average total time,

$$\langle t \rangle = \langle T_{\text{tot}} \rangle \langle t_T \rangle + \langle R_{\text{tot}} \rangle \langle t_R \rangle. \quad (\text{C6})$$

Appendix D: Statistical signature for the Chaotic and the Localized Regime

In the main text, we discuss systems featuring ballistic, chaotic and localized wave scattering, respectively. The corresponding scattering regime is determined by the number and size of impenetrable obstacles we placed inside the scattering region and can be characterized through the regime-specific transmission statistics. For the ballistic system, transmission is perfect in our case,

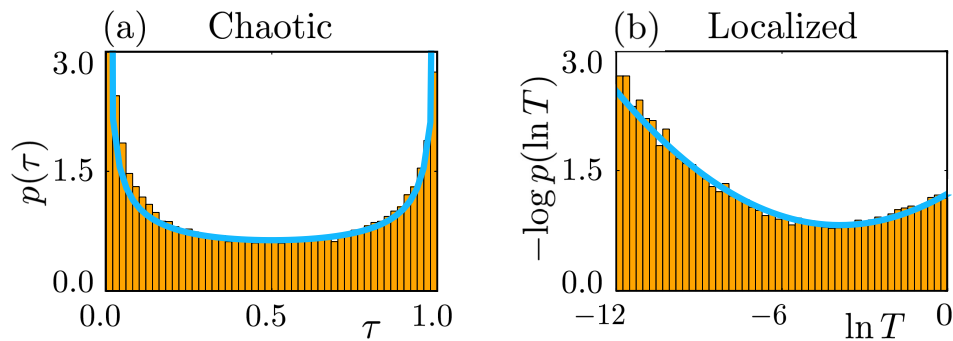


Figure 4: Transmission statistics for different transport regimes. (a) Distribution of the $t^\dagger t$ eigenvalues τ for chaotic scattering (orange bars), compared with the prediction, Eq. (D1) (light blue line). (b) Distribution of the transmission T for localized scattering compared with the prediction based on Eq. (D2). To produce the histograms, k was scanned between $k = 12.1\pi/d$ and $k = 12.9\pi/d$ and 1000 scatterer configurations were considered for each of the cases (a),(b) [in (a) only values $0.01 < \tau < 0.99$ were considered for the histogram since for very small and very large values of τ deviations from Eq. (D1) are expected [53]].

since without any scatterers we are dealing with a perfectly transmitting waveguide. In order to verify that the scattering in the systems containing a finite number of obstacles is chaotic and localized, respectively, we check whether the transmission statistics follow the respective predictions. For that purpose, we calculated the eigenvalues τ_i of the matrix $t^\dagger t$, where t is the transmission matrix. For chaotic dynamics, the τ_i follow the bimodal distribution [54–56]

$$p(\tau) = \frac{1}{\pi\sqrt{\tau(1-\tau)}}. \quad (\text{D1})$$

In a sample with Anderson localization only one single transport channel dominates the transmission [25], such that the transmission, $T = \sum_{i=1}^{N/2} \tau_i \approx \tau_{\max}$, follows the prediction for a one-dimensional wire-geometry with dis-

order [24, 25]

$$p(T) = C \frac{\sqrt{\text{arccosh}(T^{-1/2})}}{T^{3/2} (1-T)^{1/4}} \exp\left(-\frac{\xi'}{2L} \text{arccosh}^2(T^{-1/2})\right), \quad (\text{D2})$$

with C being a normalization constant. The effective localization length $\xi' = -2L/\langle \ln T \rangle$ (the brackets here mean an average over different random realizations of the positions of the hard-wall scatterers) can be determined from the numerical data. Figure 4(a),(b) shows the comparison of the numerically calculated histograms of τ and T , respectively, and their analytical predictions (D1) and (D2). We find that in both cases, the numerical data fits very well the analytical formulae, which confirms our assumptions about the scattering dynamics being chaotic or localized for the two different situations considered.

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- [1] P. Turchin, *Ecology* **72**, 1253 (1991).
 - [2] T. O. Crist and J. A. MacMahon, *Insectes Sociaux* **38**, 379 (1991).
 - [3] E. E. Holmes, *Am. Nat.* **142**, 779 (1993).
 - [4] S. Blanco and D. Fournier, *Europhys. Lett.* **61**, 168 (2003).
 - [5] A. Mazzolo, *Europhys. Lett.* **68**, 350 (2004), ISSN 1286-4854.
 - [6] O. Bénichou, M. Coppey, M. Moreau, P. H. Suet, and R. Voituriez, *Europhys. Lett.* **70**, 42 (2005), ISSN 1286-4854.
 - [7] K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wiley, Reading, Massachusetts, 1967).
 - [8] A. Campo, S. Garnier, O. Dédriché, M. Zekkri, and M. Dorigo, *PloS one* **6**, e19888 (2011), ISSN 1932-6203.
 - [9] O. Bénichou, C. Loverdo, M. Moreau, and R. Voituriez, *Phys. Chem. Chem. Phys.* **10**, 7059 (2008), ISSN 1463-9076.
 - [10] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, Cambridge, 2007).
 - [11] M. van Albada and A. Lagendijk, *Phys. Rev. Lett.* **55**, 2692 (1985).
 - [12] E. Akkermans, P. E. Wolf, and R. Maynard, *Phys. Rev. Lett.* **56**, 1471 (1986).
 - [13] A. Lagendijk, B. van Tiggelen, and D. S. Wiersma, *Phys. Today* **62**, 24 (2009).
 - [14] L. Apresyan and Y. Kravtsov, *Radiative Transfer - Statistical and Wave Aspects* (Gordon and Breach, London, 1996).
 - [15] P. Sheng, *Introduction to Wave Scattering, Localization and Mesoscopic Phenomena* (Academic Press, San Diego, 1995).
 - [16] S. Chandrasekhar, *Radiative Transfer* (Dover, New York, 1960).
 - [17] A. Lagendijk and B. van Tiggelen, *Phys. Rep.* **270**, 143 (1996).
 - [18] R. Pierrat, B. Grémaud, and D. Delande, *Phys. Rev. A* **80**, 13831 (2009).
 - [19] H. G. Winful, *Phys. Rev. Lett.* **91**, 260401 (2003).

- [20] E. Wigner, Phys. Rev. **98**, 145 (1955).
- [21] F. Smith, Phys. Rev. **118**, 349 (1960).
- [22] S. Rotter, J.-Z. Tang, L. Wirtz, J. Trost, and J. Burgdörfer, Phys. Rev. B **62**, 1950 (2000).
- [23] F. Libisch, S. Rotter, and J. Burgdörfer, **14**, 123006 (2012), ISSN 1367-2630.
- [24] V. A. Gopar and R. A. Molina, Phys. Rev. B **81**, 195415 (2010), ISSN 1098-0121.
- [25] A. Peña, A. Girschik, F. Libisch, S. Rotter, and A. A. Chabanov, Nat. comm. **5**, 3488 (2014), ISSN 2041-1723.
- [26] J. Schwinger, Phys. Rev. **82**, 664 (1951).
- [27] M. G. Krein, Sov. Math. Dokl **3**, 707 (1962).
- [28] M. G. Krein, Dokl. Akad. Nauk SSSR **144**, 475 (1962).
- [29] V. L. Lyuboshitz, Phys. Lett. **72B**, 41 (1977).
- [30] M. S. Birman and D. R. Yafaev, Algebra i Analiz **4**, 1 (1992).
- [31] G. Iannaccone, Phys. Rev. B **51**, 4727 (1995).
- [32] Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. **38**, 1918 (1997), ISSN 00222488.
- [33] S. Souma and A. Suzuki, Phys. Rev. B **65**, 115307 (2002), ISSN 0163-1829.
- [34] A. Yamilov and H. Cao, Phys. Rev. B **68**, 085111 (2003), ISSN 0163-1829.
- [35] A. Z. Genack, A. A. Chabanov, P. Sebbah, and B. A. Van Tiggelen, NATO Science Series **107**, 125 (2003).
- [36] M. Davy, Z. Shi Jing Wang, and A. Z. Genack, arXiv:1403.3811 (2014).
- [37] H. Weyl, Nachrichten der Königlichen Gesellschaft zu Göttingen pp. 110–117 (1911).
- [38] W. Arendt, R. Nittka, W. Peter, and F. Steiner, in *Mathematical Analysis of Evolution, Information, and Complexity*, edited by W. Arendt and W. P. Schleich (Wiley-VCH Verlag GmbH & Co. KGaA, 2009), pp. 1–71, ISBN 9783527628025.
- [39] H. Weyl, J. Reine Angew. Math. **143**, 177 (1913).
- [40] A. Ishimaru, *Wave propagation and scattering in random media*, vol. 2 (Academic press New York, 1978).
- [41] K. Vynck, M. Burrelli, F. Riboli, and D. S. Wiersma, Nat Mater **11**, 1017 (2012).
- [42] E. Yablonovitch and G. D. Cody, Electron Devices, IEEE Transactions on **29**, 300 (1982).
- [43] M. V. Berry, *Distribution of Modes in Fractal Resonators* (Springer Science + Business Media, 1979), pp. 51–53.
- [44] M. L. Lapidus, *Can One Hear the Shape of a Fractal Drum? Partial Resolution of the Weyl-Berry Conjecture* (Springer Science + Business Media, 1991), pp. 119–126.
- [45] H. Cao, Waves in random media **13**, R1 (2003).
- [46] Y. Chong and A. D. Stone, Physical review letters **107**, 163901 (2011).
- [47] A. P. Mosk, A. Lagendijk, G. Leroosey, and M. Fink, Nature photonics **6**, 283 (2012).
- [48] J. Aulbach, B. Gjonaj, P. M. Johnson, A. P. Mosk, and A. Lagendijk, Physical review letters **106**, 103901 (2011).
- [49] I. Vellekoop and A. Mosk, Physical review letters **101**, 120601 (2008).
- [50] S. Rotter, P. Ambichl, and F. Libisch, Phys. Rev. Lett. **106**, 120602 (2011).
- [51] P. Barthelemy, J. Bertolotti, and D. S. Wiersma, Nature **453**, 495 (2008).
- [52] D. S. Wiersma, Nature Photonics **7**, 188 (2013).
- [53] S. Rotter, F. Aigner, and J. Burgdörfer, Phys. Rev. B **75**, 125312 (2007).
- [54] H. U. Baranger and P. A. Mello, Phys. Rev. Lett. **73**, 142 (1994).
- [55] R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhys. Lett. **27**, 255 (1994), ISSN 0295-5075.
- [56] C. Beenakker, Rev. Mod. Phys. **69**, 731 (1997), ISSN 0034-6861.
- [57] V. Sokolov and V. Zelevinsky, Phys. Rev. C **56**, 311 (1997), ISSN 0556-2813.
- [58] The International System of Units is used throughout this article.
- [59] One can show that the quantity measured by the Wigner-Smith time-delay operator is equal to the dwell time (Eq. (3)) if the frequency dependence of the coupling between the scattering region and its surrounding becomes negligible [57]. This is the case in the systems considered here.